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LETTER TO THE EDITOR

On q-squeezed states

Allan I Solomon† and Jacob Katriel

Department of Chemistry, Technion-Israel Institute of Technology, Haifa 32000, Israel

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Abstract. We present the results of an analysis of the squeezing of components of the (conventional) electromagnetic field in quantum group analogues of the Heisenberg–Weyl (HW) coherent state and SU(1, 1) squeezed state. We find that squeezing occurs for all finite-q values not equal to unity in the HW q-coherent state, in contrast to the usual case; and also in the $SU_q(1,1)$ case, although here less than in the usual ($q = 1$) SU(1, 1) squeezed state.

Recent interest in the so-called quantum groups has led to several papers which describe explicitly q-analogues of boson operators [1–3]. Using these operators it is possible to construct q-analogues of coherent states and it is natural to investigate the squeezing properties of such states; i.e. to what extent these states reduce uncertainty expectations of components of the electromagnetic field below their vacuum values. The answer is well known in the conventional (Glauber) coherent case [4]; there is no squeezing. One has to introduce *squeezed* states—essentially coherent states of the SU(1, 1) group—in order to obtain this indeterminacy reduction effect. In this letter we present the rather surprising result that for q-coherent states squeezing occurs for all q values (other than the 'classical' limit $q = 1$). We further present the analogous result for the $SU_q(1,1)$ quantum group, *en route* showing that a naive analogue of the q-coherent state definition fails in this case.

We first of all define q-boson operators a, $a^\dagger$ as in [1–3], to which we refer the reader for further motivation of the definition. We start with conventional bosons b, $b^\dagger$ satisfying $[b, b^\dagger] = I$, $b^\dagger b = N$ and write [3]

$$a = \left(\frac{[N+1]}{N+1}\right)^{1/2} b \qquad a^\dagger = b^\dagger \left(\frac{[N+1]}{N+1}\right)^{1/2} \tag{1}$$

where

$$[x] \equiv [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2}$$

The operators a, $a^\dagger$ satisfy, on the usual Fock space,

$$a|0\rangle = 0 \qquad |n\rangle = ([n]!)^{-1/2}(a^\dagger)^n|0\rangle \tag{3}$$

where $[n]! \equiv [n][n-1]\ldots[1]$ by a useful abuse of notation. It is convenient to define $[0]! \equiv 1$ and also $[n]!!$ in the obvious way. The conventional Weyl–Heisenberg coherent

† Permanent address: Faculty of Mathematics, The Open University, Milton Keynes MK7 6AA, UK.

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state $\exp(\alpha a^\dagger)|0\rangle$ is not available for our q-operators here since it fails to be normalizable, and thus also in the Hilbert space, for all $q \neq 1$ (non-zero α). Instead, use of the alternatively available conventional definition $a|\alpha\rangle = \alpha|\alpha\rangle$ yields [2, 3]

$$|\alpha\rangle = \mathcal{N}^{-1} \exp_q(\alpha a^\dagger)|0\rangle \tag{4}$$

with

$$\mathcal{N}^2 = \exp_q(|\alpha|^2) \tag{5}$$

where

$$\exp_q(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]!}. \tag{6}$$

It is with respect to this state $|\alpha\rangle$ that we calculate the dispersions of the electromagnetic field, assumed to be expressed in terms of the conventional operators b, $b^\dagger$ in the standard way

$$x = (b + b^\dagger)/\sqrt{2} \qquad p = (b - b^\dagger)/(\mathrm{i}\sqrt{2}). \tag{7}$$

Thus,

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 \qquad (\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2$$

so that

$$(\Delta x)^2 = \tfrac{1}{2} + \langle b^\dagger b\rangle - \langle b^\dagger\rangle\langle b\rangle + \mathrm{Re}\{\langle (b^\dagger)^2\rangle - \langle b^\dagger\rangle^2\} \tag{8}$$

and

$$(\Delta p)^2 = \tfrac{1}{2} + \langle b^\dagger b\rangle - \langle b^\dagger\rangle\langle b\rangle - \mathrm{Re}\{\langle (b^\dagger)^2\rangle - \langle b^\dagger\rangle^2\}. \tag{9}$$

Note that the expectations in (8) and (9) are all with respect to the q-coherent state (4). The results for various values of the parameters α and q are presented in figure 1. As expected, the 'classical' limit $q = 1$ yields no squeezing (that is, the value of $(\Delta x)^2$ is not reduced below its vacuum value of $\frac{1}{2}$). The asymptotic limit $q \to \infty$ for

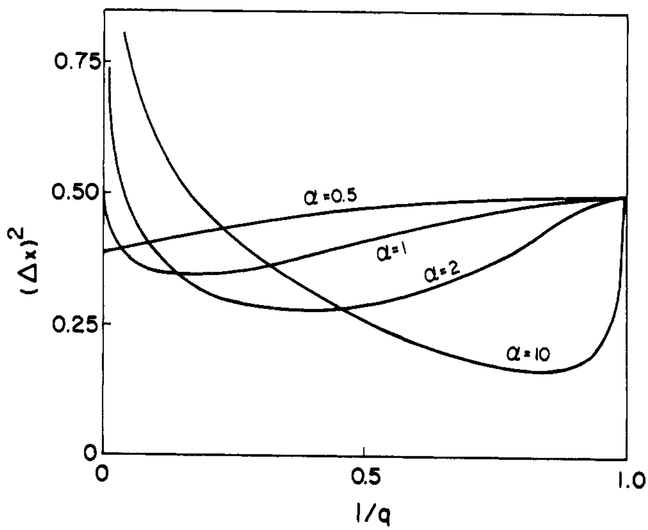

Figure 1. Squeezing in the HW q-coherent state ($q = 1$ corresponds to the conventional coherent state).

$(\Delta x)^2$ is $\frac{1}{2}+\alpha^2(\alpha^2-1)/(\alpha^2+1)^2$. The analysis is symmetric under the interchange $q \leftrightarrow 1/q$. Note that for three of the values of α displayed in figure 1 there is a finite value of q giving optimal squeezing; in fact, the asymptotics would imply that there is an optimal value for each $\alpha > 1/\sqrt{3}$.

Conventional squeezed states are obtained as the group coherent states of SU(1, 1), whose Lie algebra has relations

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0. \tag{10}$$

They have typically the form $\exp(\frac{1}{2}\alpha(a^\dagger)^2)|0\rangle$. The corresponding algebra of the quantum group $SU_q(1, 1)$ may be written

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -[2K_0]_{q^2} \tag{11}$$

with realization

$$K_0 = \tfrac{1}{2}(N+\tfrac{1}{2}) \qquad K_+ = k(a^\dagger)^2 \qquad K_- = ka^2 \qquad (k=(q+q^{-1})^{-1})$$

in terms of the q-bosons a, $a^\dagger$ [3].

In this case the realization in terms of the ordinary exponential function fails to give a normalizable state, as was the case for the Heisenberg-Weyl coherent states (for all $q \neq 1$). Further, the modified exponential form $\exp_q(\frac{1}{2}\alpha(a^\dagger)^2)$ also fails to give a normalizable state for $q \neq 1$. However, noting that the conventional squeezed state $|\alpha\rangle \equiv \exp(\frac{1}{2}\alpha(a^\dagger)^2)|0\rangle$ satisfies

$$(a - \alpha a^\dagger)|\alpha\rangle = 0 \tag{12}$$

we may use (12) as our definition of a q-squeezed state corresponding to $SU_q(1, 1)$, to obtain

$$|\alpha\rangle = \mathcal{N}^{-1} \sum^{\infty} \alpha^n \sqrt{\frac{[2n-1]!!}{[2n]!!}}|2n\rangle \tag{13}$$

with normalization

$$\mathcal{N}^2 = \sum^{\infty} |\alpha|^{2n} \frac{[2n-1]!!}{[2n]!!}. \tag{14}$$

The results of numerical computations of the dispersions (8) and (9), now taken with respect to the state (13), for various values of the squeezing parameter α and quantum parameter q, are presented in figure 2. Again we note that in the conventional

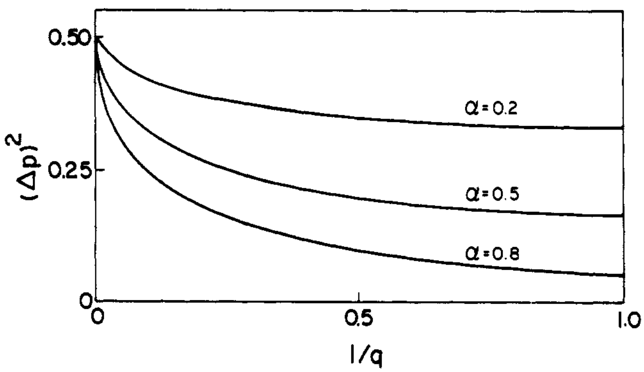

Figure 2. Squeezing in the $SU_q(1, 1)$ squeezed state ($q = 1$ corresponds to the conventional squeezed state).

limit $q=1$ we obtain the standard SU(1, 1) squeezing values (see for example [5]). Unlike the q-coherent case, here the squeezing obtained is always less than that obtained in the conventional case, represented by the minima of the curves on the $q=1$ ordinate. In the asymptotic limit $q \to \infty$ we obtain the vacuum values $(\Delta x)^2 = (\Delta p)^2 = \frac{1}{2}$. Generalizations of these results to the multiphoton q-states will be presented subsequently.

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